Patterns and Connections
in
Developmental Mathematics

Pat McKeague
1339 Marsh Street
San Luis Obispo, CA 93401
(805) 541-4593
pat@mckeague.com

Transparencies:
www.mckeague.com

Number Sequences
and
Inductive Reasoning

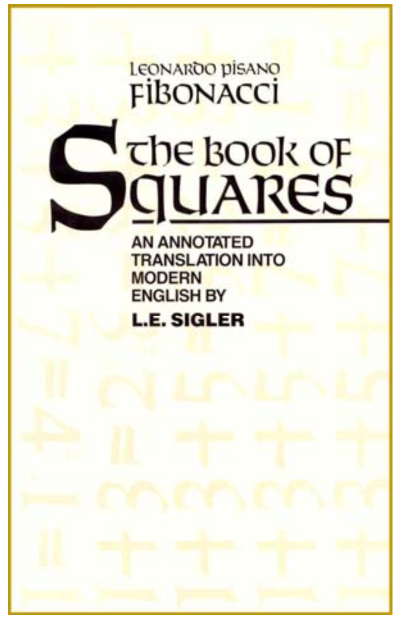
Sequence of Odd Numbers

1, 3, 5, 7, . . .

Sequence of Squares

1, 4, 9, 16, . . .

Introduction



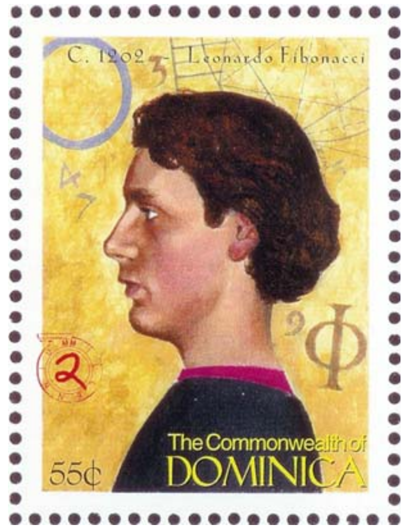
I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbers; for the unity is a square and from it is made the first square, namely 1; to this unity is added 3, making the second square, namely 4, with root 2; if to the sum is added the third odd number, namely 5, the third square is created, namely 9, with root 3; and thus sums of consecutive odd numbers and a sequence of squares always arise together in order.

Odds: 1, 3, 5, 7, . . .

Squares: 1, 4, 9, 16, . . .

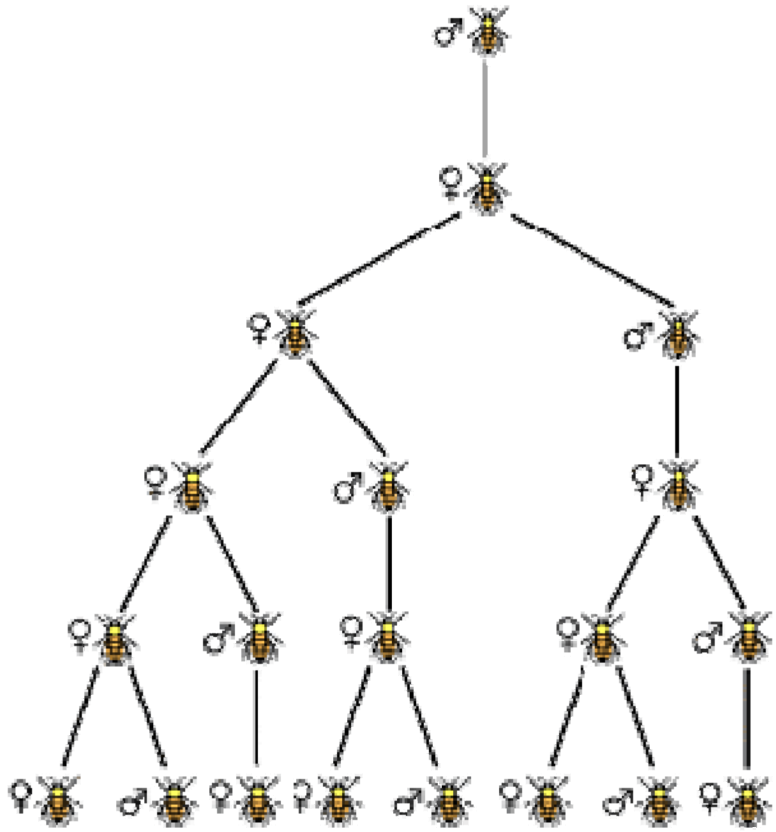
$$
\begin{aligned}
1 &= 1 = 1^2 \\
1 + 3 &= 4 = 2^2 \\
1 + 3 + 5 &= 9 = 3^2 \\
1 + 3 + 5 + 7 &= 16 = 4^2 \\
1 + 3 + 5 + 7 + 9 &= 25 = 5^2
\end{aligned}
$$

Fibonacci Sequence

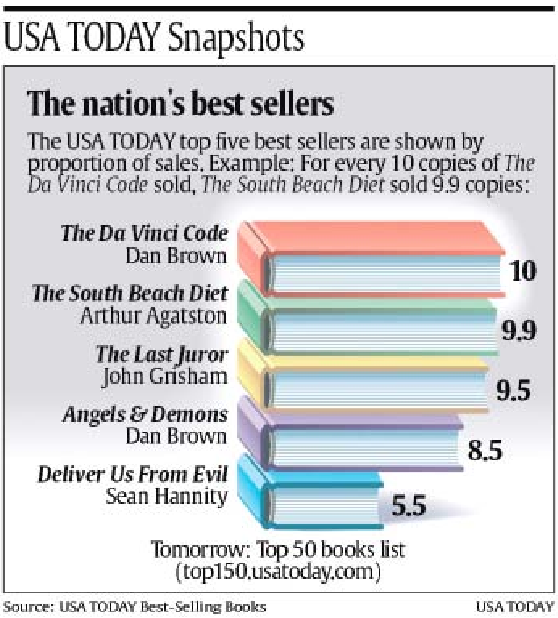


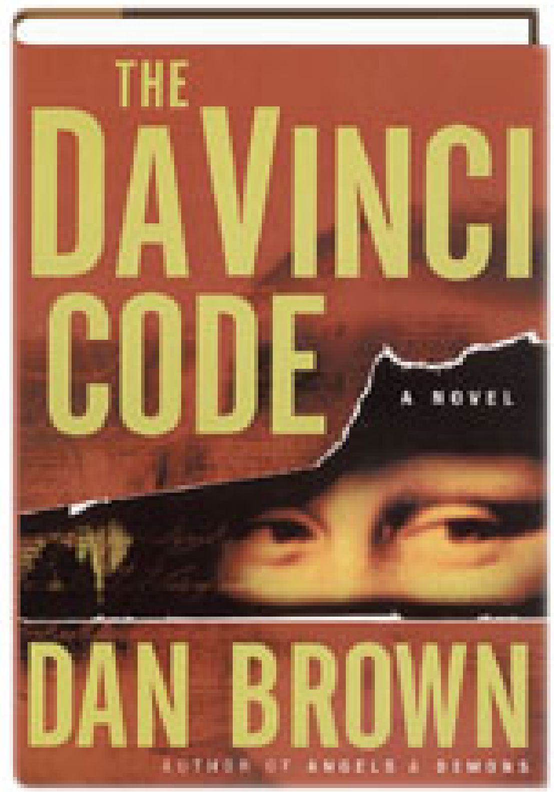
1, 1, 2, 3, 5, 8, . . .

Application: Count the number of bees in each generation of the family tree of a male honey bee. (A male honey bee has only one parent, a mother. A female honey bee has two parents, a mother and a father.)



Mathematics around Us

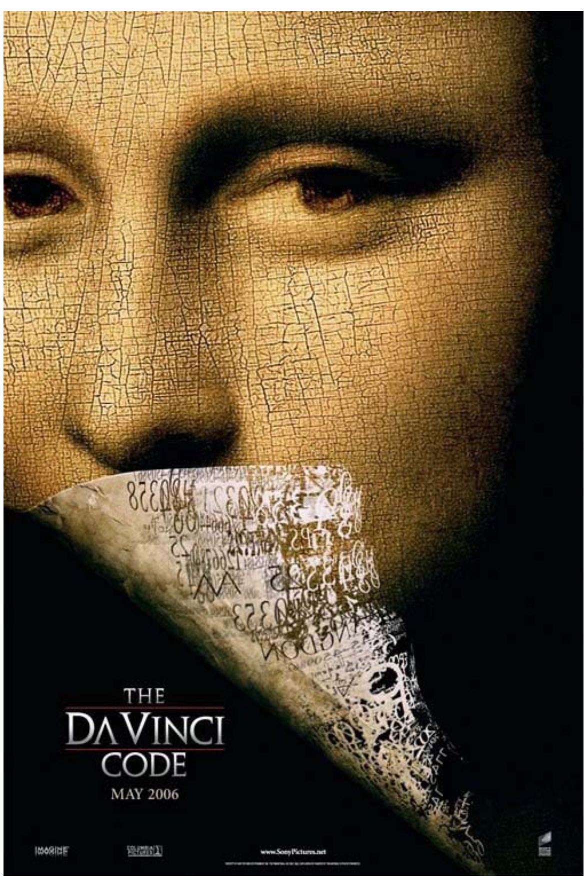




13-3-2-21-1-1-8-5

O Draconian devil!

Oh, lame saint!



Merriam-Webster's COLLEGIATE® DICTIONARY

Atlas | Reverse Dictionary | Rhyming Dictionary

Collegiate® Dictionary | Collegiate® Thesaurus | Unabridged

Fibonacci number

One entry found for **Fibonacci number**.
Main Entry: **Fi·bo·nac·ci number**
Pronunciation: "fE-b&-'nä-chE-, "fi-b&-
Function: *noun*
Etymology: Leonardo *Fibonacci* died *ab* 1250 Italian mathematician
: an integer in the infinite sequence 1, 1, 2, 3, 5, 8, 13, ... of which the first two terms are 1 and 1 and each succeeding term is the sum of the two immediately preceding

W. H. FREEMAN AND COMPANY San Francisco

§3. 곱셈공식

다항식의 곱셈을 하는 데, 앞 절의 공식이 기초가 된다. 그런데 특수한 모양의 곱셈은 따로 공식을 만들어서, 그것을 활용하면 계산이 편하다. 여기서 이러한 곱셈공식과 활용법을 연구하자.

곱셈공식 1

$(a+b)^2=a^2+2ab+b^2$

(합의 제곱)

곱셈공식 2

$(a-b)^2=a^2-2ab+b^2$

(차의 제곱)

풀이 〔1〕 $(a+b)^2$

$=(a+b)(a+b)$

$=a^2+ab+ab+b^2$

$=a^2+2ab+b^2$

〔2〕

$$\begin{array}{r} a+b \\ \times)\ a+b \\ \hline a^2+ab \quad \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

풀이 〔1〕 $(a-b)^2$

$=(a-b)(a-b)$

$=a^2-ab-ab+b^2$

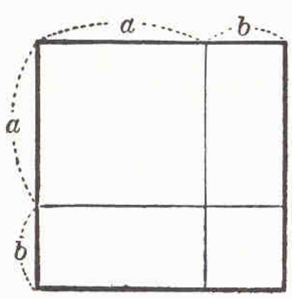
$=a^2-2ab+b^2$

〔2〕

$$\begin{array}{r} a-b \\ \times)\ a-b \\ \hline a^2-ab \quad \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

〖주의〗 $(a+b)^2$을 a^2+b^2, $(a-b)^2$을 a^2-b^2으로 하지 않도록 주의를 해야 한다.

물음 1. 오른편 그림을 보고 면적 관계를 이용하여, 합의 제곱공식이 성립함을 증명하여라.



TRANSPARENCY 10-19

To be used with ELEMENTARY ALGEBRA by Harold R. Jacobs

BINOMIAL EXPANSIONS

$(x + y)^0 =$ $\mathbf{1}$

$(x + y)^1 =$ $x + y$

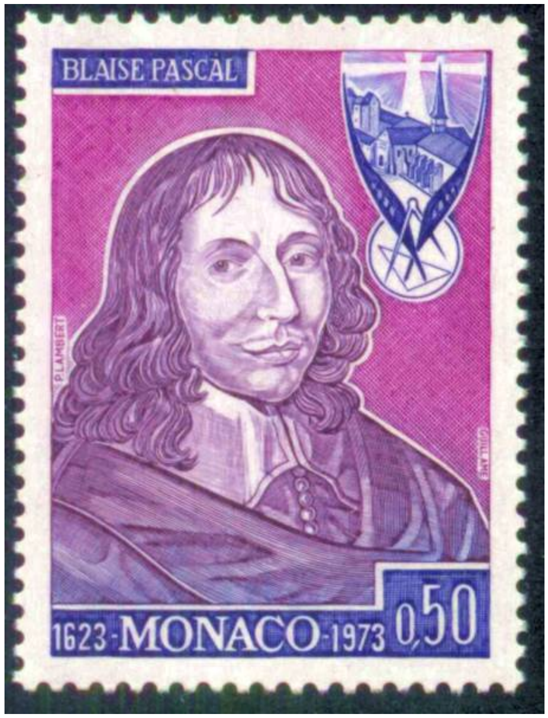
$(x + y)^2 =$ $x^2 + \mathbf{2}xy + y^2$

$(x + y)^3 =$ $x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + y^3$

$(x + y)^4 =$ $x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + y^4$

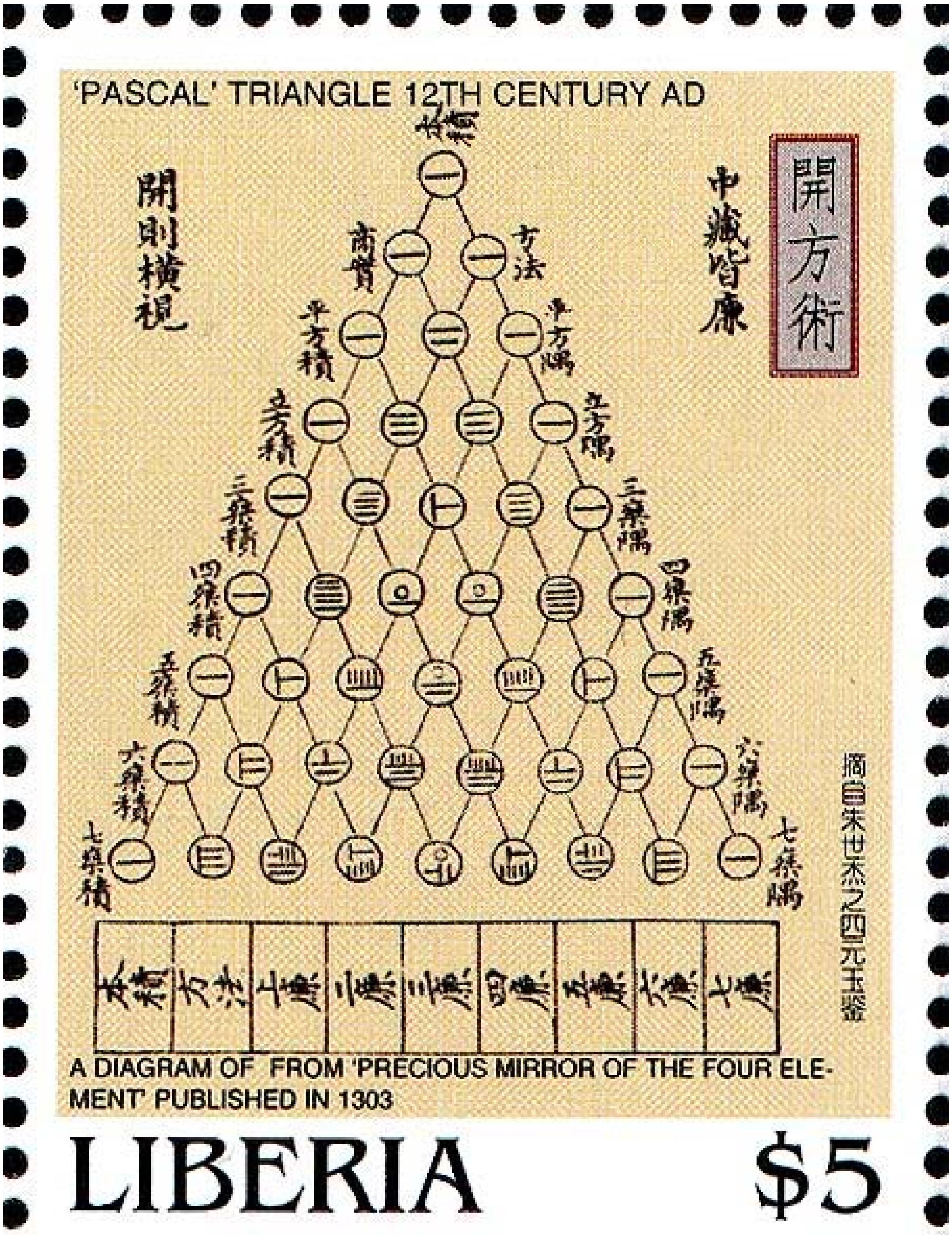
$(x + y)^5 =$ $x^5 + \mathbf{5}x^4y + \mathbf{10}x^3y^2 + \mathbf{10}x^2y^3 + \mathbf{5}xy^4 + y^5$

$(x + y)^6 =$ $x^6 + \mathbf{6}x^5y + \mathbf{15}x^4y^2 + \mathbf{20}x^3y^3 + \mathbf{15}x^2y^4 + \mathbf{6}xy^5 + y^6$



Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1


'PASCAL' TRIANGLE 12TH CENTURY AD
開方術
A DIAGRAM OF FROM 'PRECIOUS MIRROR OF THE FOUR ELE-
MENT' PUBLISHED IN 1303
LIBERIA
$5

The Connection Between Pascal's Triangle and the Fibonacci Sequence

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

A Complex Fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}$$

A Continued Fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

Merriam-Webster's COLLEGIATE® DICTIONARY

Atlas | Reverse Dictionary | Rhyming Dictionary

Collegiate® Dictionary | Collegiate® Thesaurus | Unabridged

continued fraction

One entry found for **continued fraction**.
Main Entry: **continued fraction**
Function: *noun*
: a fraction whose numerator is an integer and whose denominator is an integer plus a fraction whose numerator is an integer and whose denominator is an integer plus a fraction and so on

Working with Continued Fractions

$$1+\frac{1}{1+1}$$

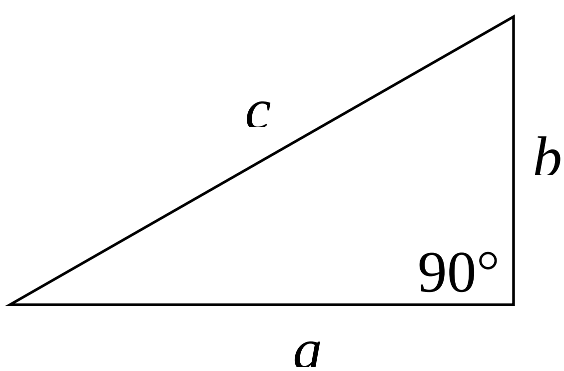
$$1+\frac{1}{1+\frac{1}{1+1}}$$

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$$

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}$$

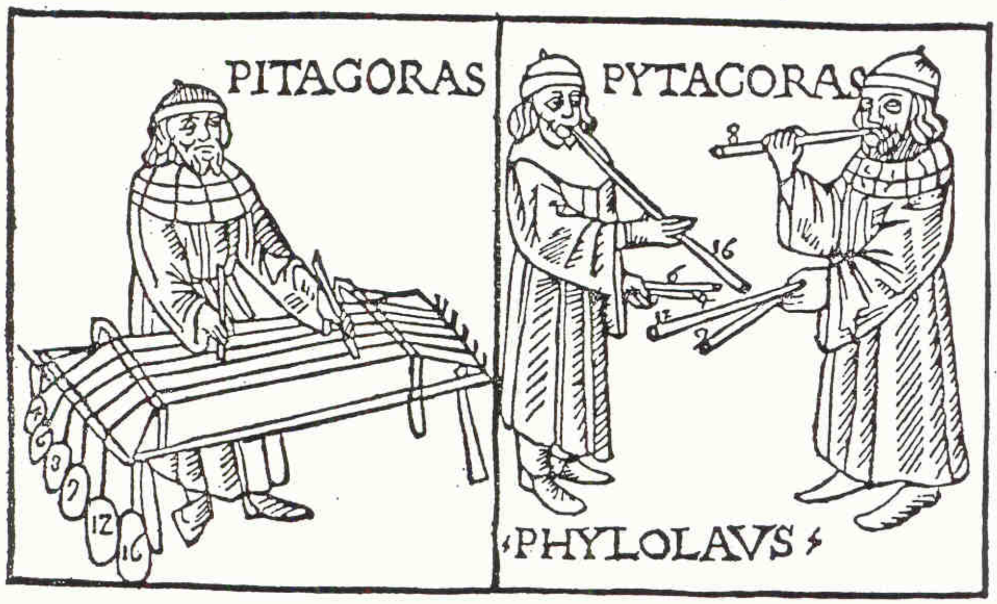
Pythagorean Theorem

In any right triangle, the square of the longest side (the hypotenuse) is equal to the sum of the squares of the other two sides (the legs).



$$c^2 = a^2 + b^2$$

Pythagoras and Music



PYTHAGORAS THE MUSICIAN

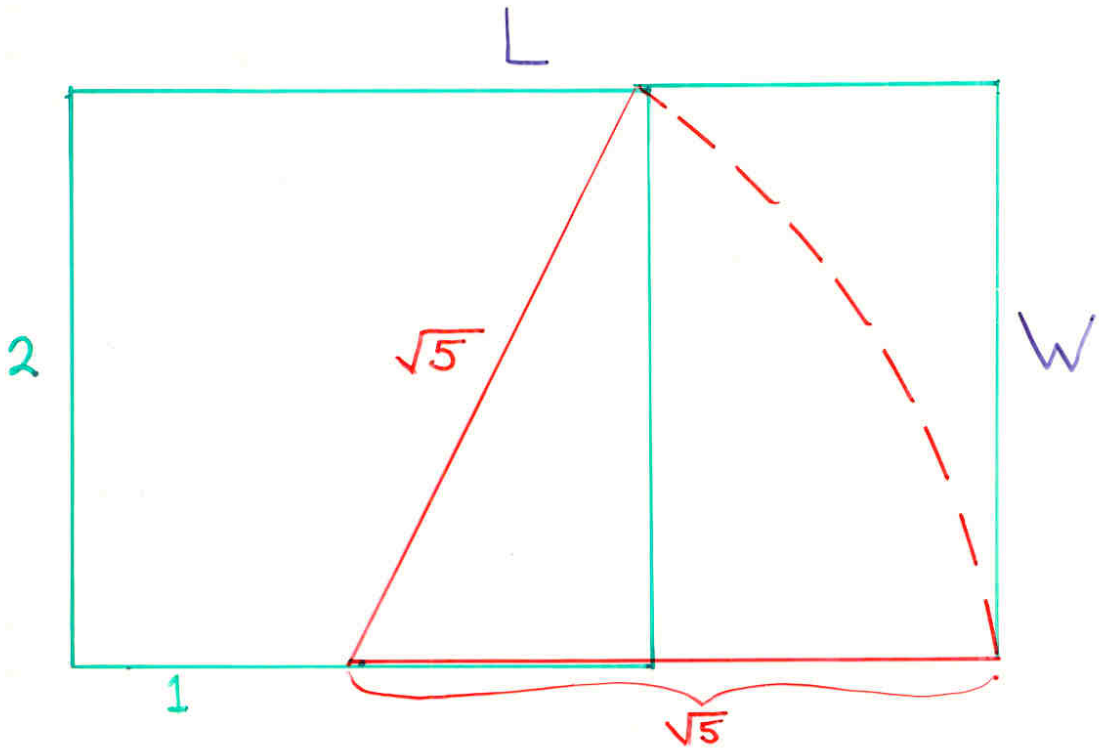
From F. Gafurius, *Theorica Musice*, Milan, 1492. One of the first crude attempts to portray Pythagoras by means of a woodcut, and the first to portray him as a musician.

Pythagoras and Shakespeare

Thou almost mak’st me waver in my faith,
To hold opinion with Pythagoras,
That souls of animals infuse themselves
Into the trunks of men.

Merchant of Venice

An Application of the Pythagorean Theorem: The Golden Rectangle



Golden Ratio $= \frac{L}{W} = \frac{1+\sqrt{5}}{2}$

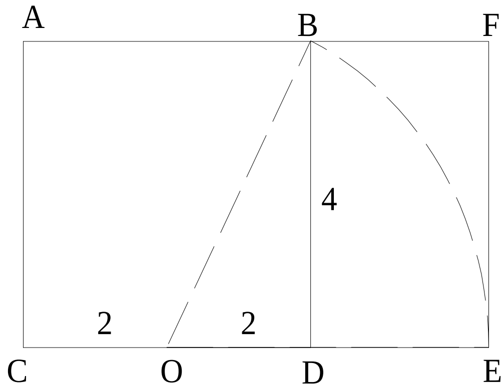
≈ 1.618

Similar Figures/Simplifying Roots

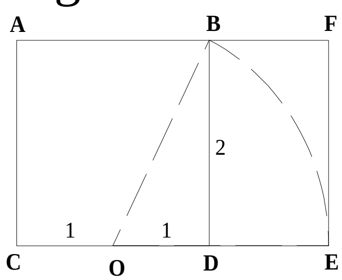
Similar Figures have the same shape, but not necessarily the same size.

For each figure below, first find the length of the diagonal OB. Then find an expression for the ratio CE/EF. Convert each ratio to a decimal.

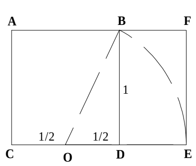
A golden rectangle from a square of side 4.



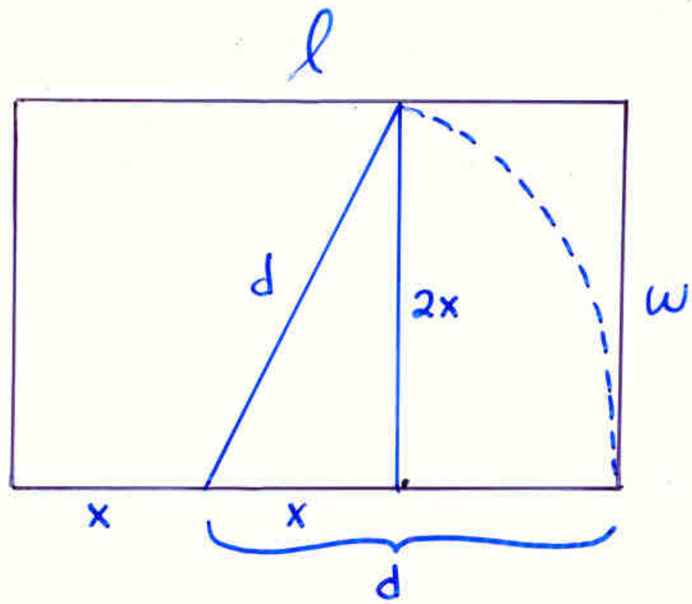
A golden rectangle from a square of side 2.



A golden rectangle from a square of side 1.



All Golden Rectangles are Similar



$$d = \sqrt{x^2 + (2x)^2}$$

$$= \sqrt{x^2 + 4x^2}$$

$$= \sqrt{5x^2}$$

$$= x\sqrt{5}$$

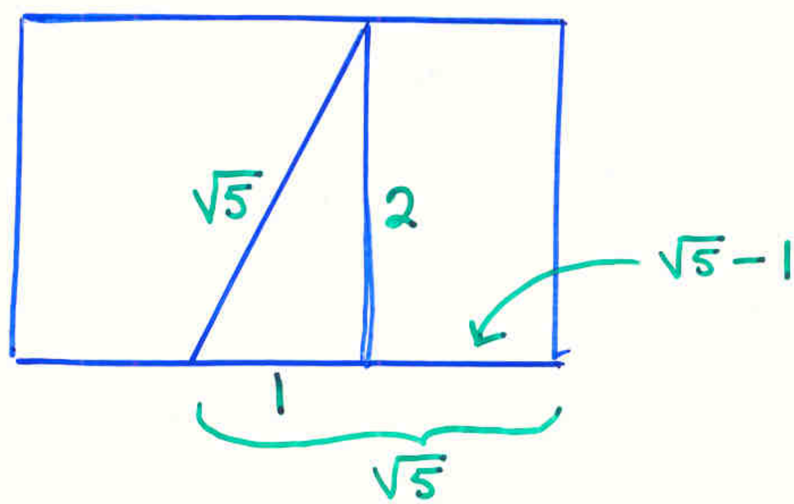
$$\frac{\ell}{w} = \frac{x + x\sqrt{5}}{2x}$$

$$= \frac{x(1+\sqrt{5})}{2x}$$

$$= \frac{1+\sqrt{5}}{2}$$

Rationalizing the Denominator

The Golden Rectangle Contains Another Golden Rectangle



Small Rectangle

$$\frac{l}{w} = \frac{2}{\sqrt{5}-1} = \frac{2}{\sqrt{5}-1}\cdot\frac{\sqrt{5}+1}{\sqrt{5}+1}$$

$$= \frac{2\sqrt{5}+2}{5-1}$$

$$= \frac{2+2\sqrt{5}}{4}$$

$$= \frac{2(1+\sqrt{5})}{2\cdot 2}$$

$$= \frac{1+\sqrt{5}}{2}$$

Al-Samawal

Born: about 1130 in Baghdad, Iraq
Died: about 1180 in Maragha, Iran

The golden ratio is $\frac{\sqrt{125}-5}{15-\sqrt{125}}$

$$\frac{\sqrt{125}-5}{15-\sqrt{125}} = \frac{5\sqrt{5}-5}{15-5\sqrt{5}}$$

$$= \frac{5(\sqrt{5}-1)}{5(3-\sqrt{5})}$$

$$= \frac{\sqrt{5}-1}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$= \frac{3\sqrt{5}+5-3-\sqrt{5}}{9-3\sqrt{5}+3\sqrt{5}-5}$$

$$= \frac{2\sqrt{5}+2}{4}$$

$$= \frac{2(\sqrt{5}+1)}{2 \cdot 2}$$

$$= \frac{1+\sqrt{5}}{2}$$

A Surprising Relationship

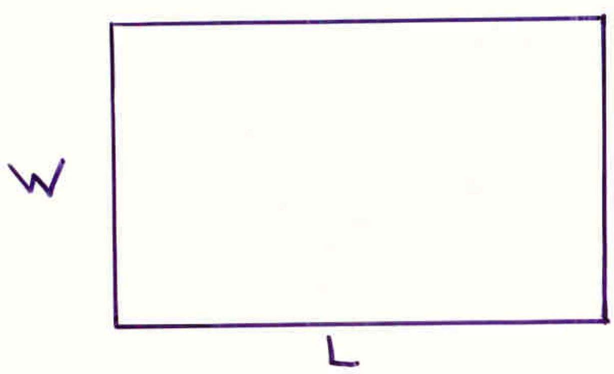
A Continued Fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The Fibonacci Sequence

1, 1, 2, 3, 5, 8, ...

The Golden Rectangle



$$\frac{L}{W} = \frac{1+\sqrt{5}}{2}$$

$$\approx 1.618$$

Finding the Relationship using Inductive Reasoning

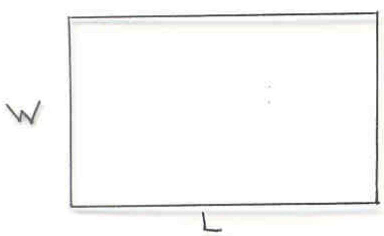
A Continued Fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

The Fibonacci Sequence

1, 1, 2, 3, 5, 8, . . .

The Golden Rectangle



$$\frac{L}{W} = \frac{1+\sqrt{5}}{2}$$

$$\approx 1.618$$

A Sequence of Complex Fractions

$$1+\cfrac{1}{1+1} = \frac{3}{2} = 1.5000$$

$$1+\cfrac{1}{1+\cfrac{1}{1+1}} = \frac{5}{3} = 1.6667$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}} = \frac{8}{5} = 1.6000$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}} = \frac{13}{8} = 1.6250$$

$$= \frac{21}{13} = 1.6153$$

$$= \frac{34}{21} = 1.6191$$

$$= \frac{55}{34} = 1.6176$$

$$= \frac{89}{55} = 1.6182$$

$$= \frac{144}{89} = 1.6180$$

therefore,

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{1+\sqrt{5}}{2}$$

The Quadratic Formula in French

II. Équations et inéquations du second degré

9.7 RÉSOLUTION DE L'ÉQUATION DU SECOND DEGRÉ

Théorème.

Étant donné l'équation

(1) $$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

Si $\Delta = b^2 - 4\,ac > 0$, l'équation (1) a **deux racines distinctes**

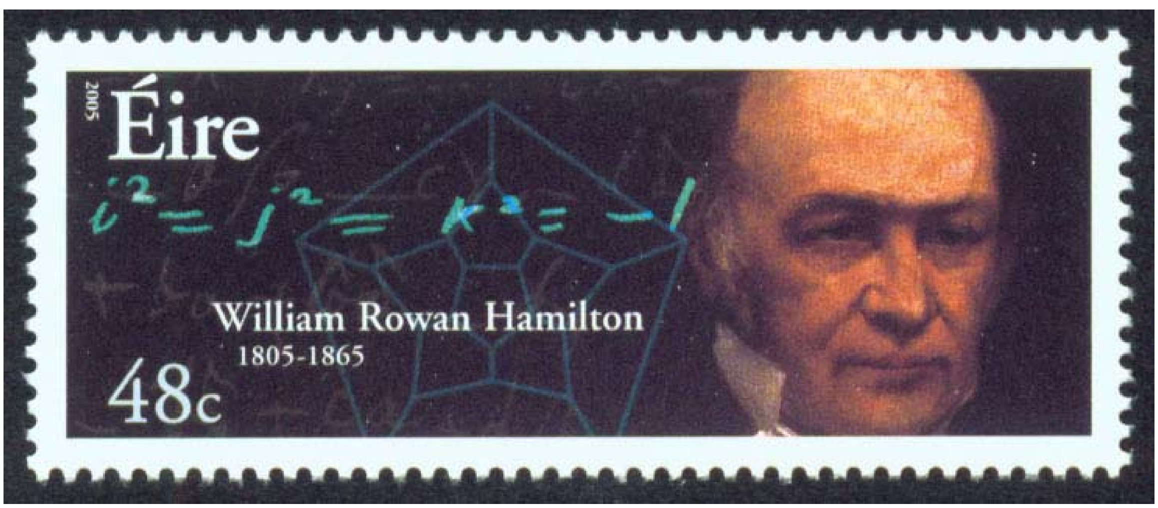
$$\alpha = \frac{-b - \sqrt{b^2 - 4\,ac}}{2a}, \qquad \beta = \frac{-b + \sqrt{b^2 - 4\,ac}}{2\,a};$$

Si $\Delta = b^2 - 4\,ac = 0$, l'équation (1) a une **racine double**

$$\alpha = \beta = \frac{-b}{2\,a};$$

Si $\Delta = b^2 - 4\,ac < 0$ l'équation (1) n'a **pas de racine** dans $\mathbb{R}$.

Complex Numbers on an Irish Postage Stamp



Showing the Relationship with Deductive Reasoning

If $$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}$$

then $$x = 1 + \frac{1}{x}$$

$$\Rightarrow \quad x^2 = x + 1$$

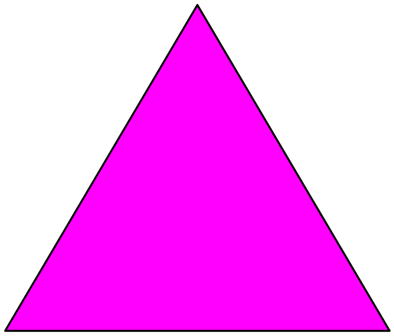
$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$= \frac{1 + \sqrt{5}}{2}$$

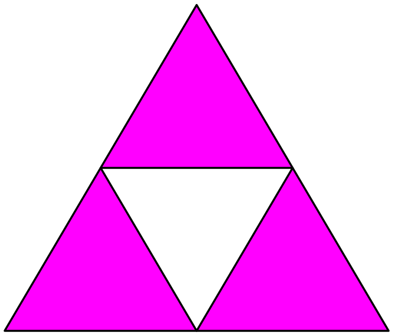
Sierpinski Triangle

If the area of the shaded region in Stage 0 is 1, find the area of the shaded region in the other stages.



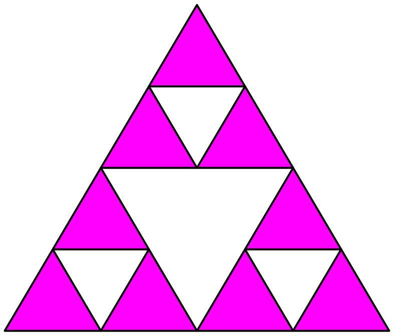
Stage 0

Area = 1



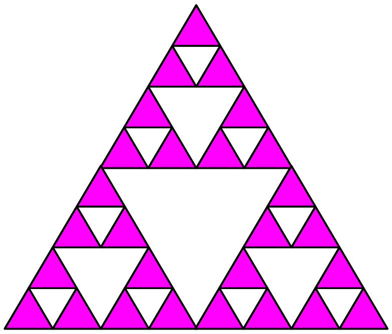
Stage 1

Area =



Stage 2

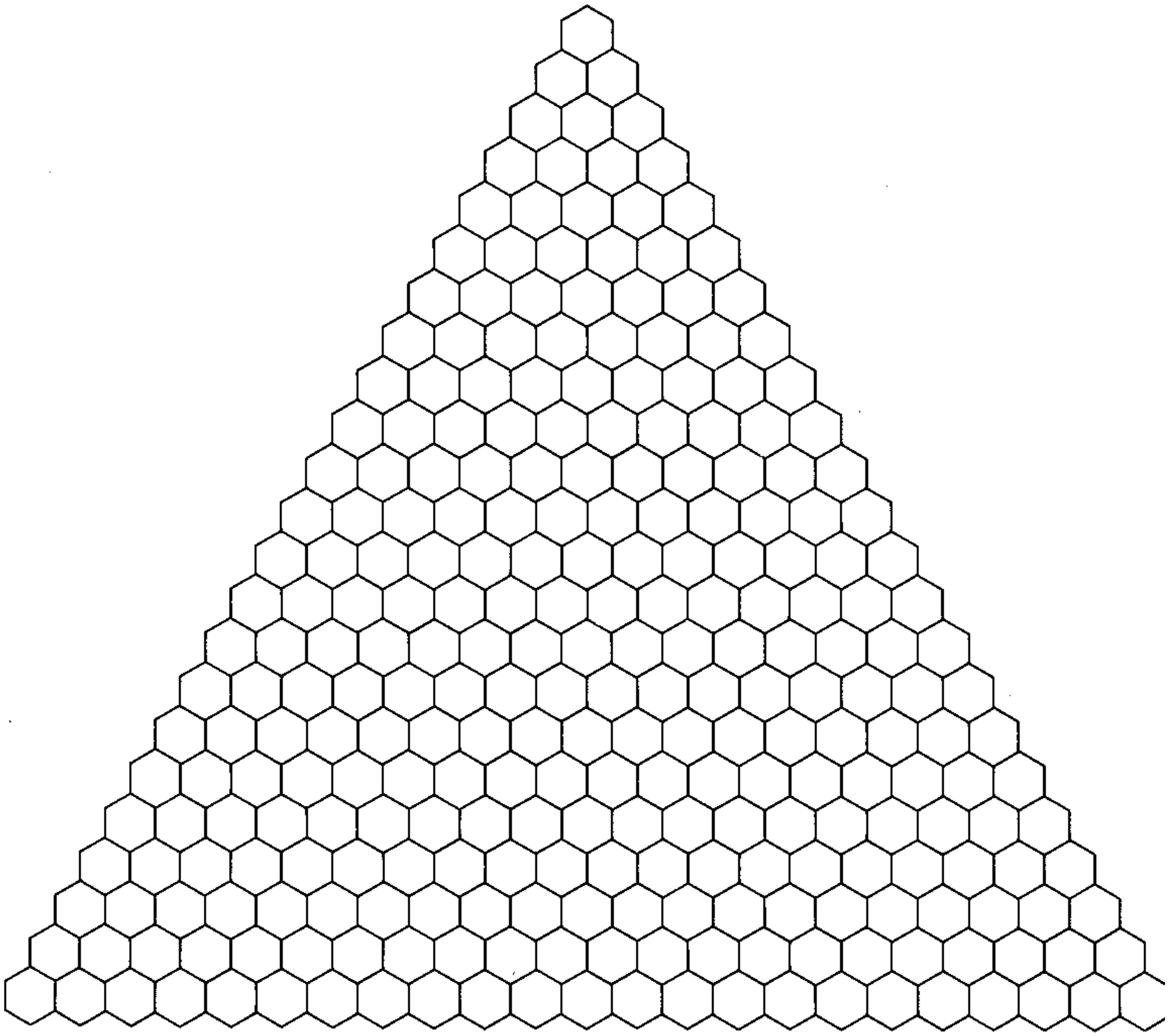
Area =

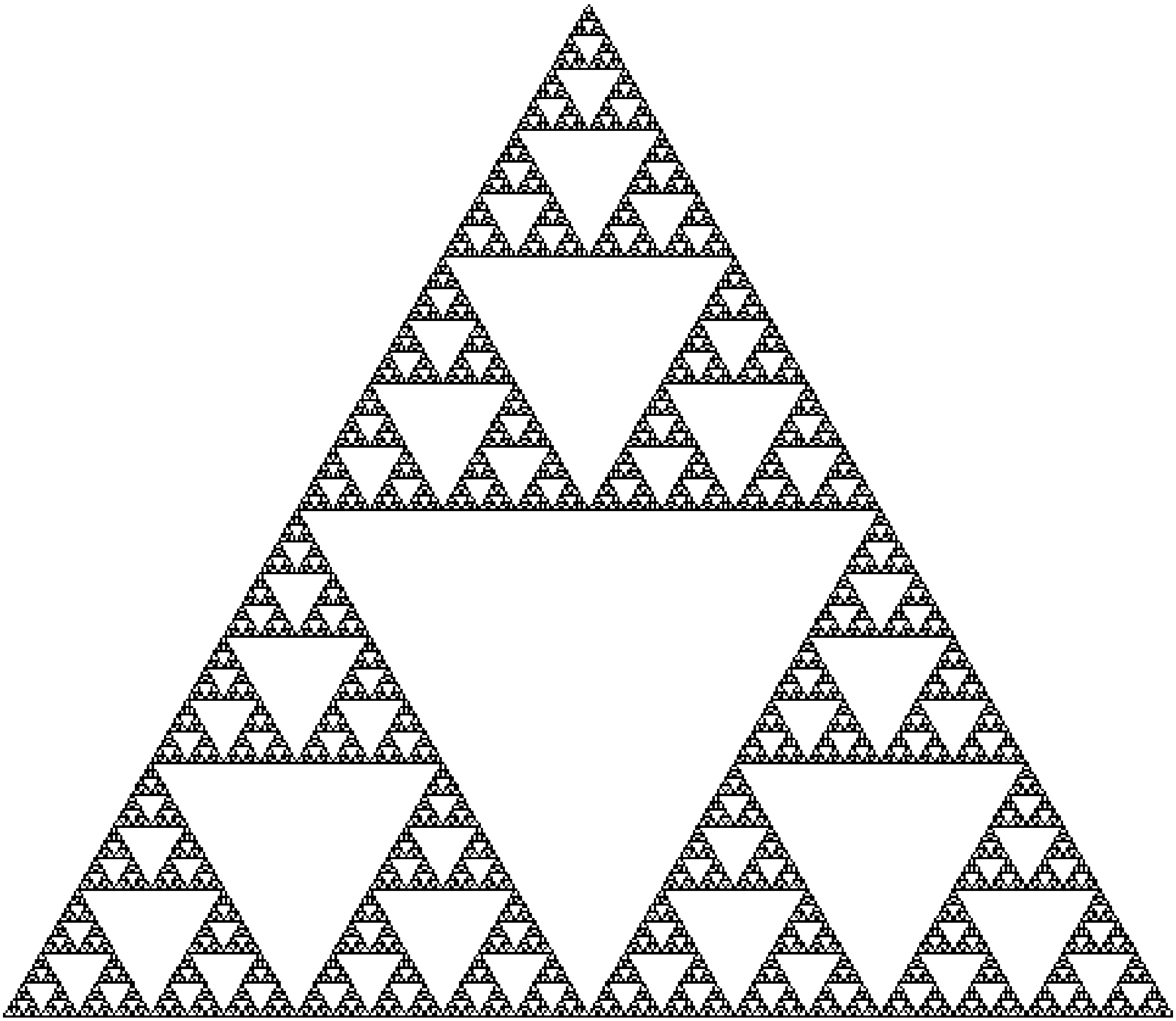


Stage 3

Area =

Group Project: Fill in using numbers from Pascal's Triangle, then color in each polygon that contains an odd number.

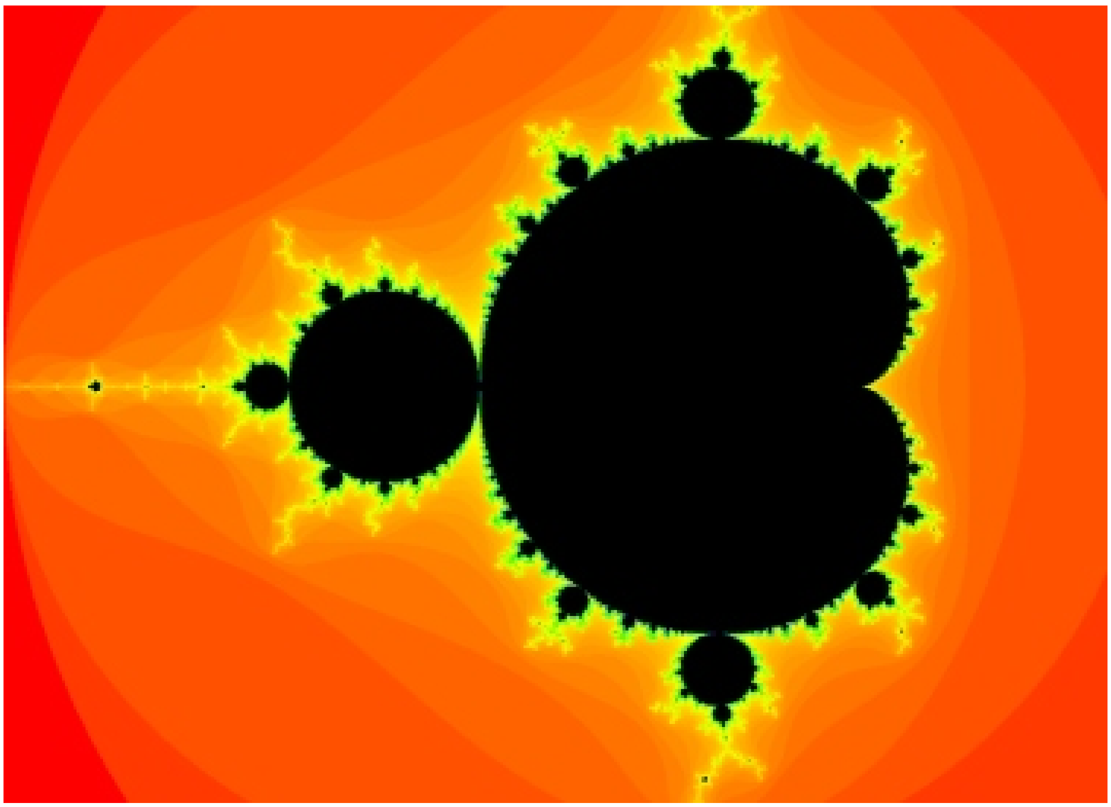




Sierpinski Pyramid

Postage stamp issued by Hungary in 1996 on the occasion of a Mathematical Congress held in Hungary.

Mandelbrot Set



Fractal Landscape

Merriam-Webster's COLLEGIATE® DICTIONARY

Atlas | Reverse Dictionary | Rhyming Dictionary

Collegiate® Dictionary | Collegiate® Thesaurus | Unabridged

One entry found for **fractal**.

Main Entry: **frac·tal**
Pronunciation: 'frak-t&l
Function: *noun*
Etymology: French *fractale,* from Latin *fract*us broken, uneven (past participle of *frangere* to break) + French *-ale* -al (noun suffix)
Date: 1975
: any of various extremely irregular curves or shapes for which any suitably chosen part is similar in shape to a given larger or smaller part when magnified or reduced to the same size
- fractal *adjective*

Iterated Functions

Definition: If x is in the domain of a function f, then the sequence below is called an *orbit* of x for f. Each member of the sequence is called an *iterate* of x for f.

$$x, \quad f(x), \quad f(f(x)), \quad f(f(f(x))), \; . \; . \; .$$

A Regular Function

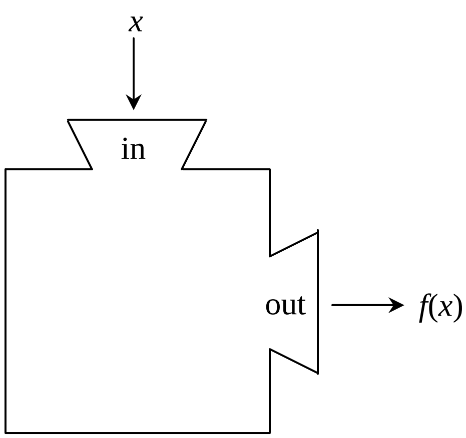


Figure 1

An Iterated Function

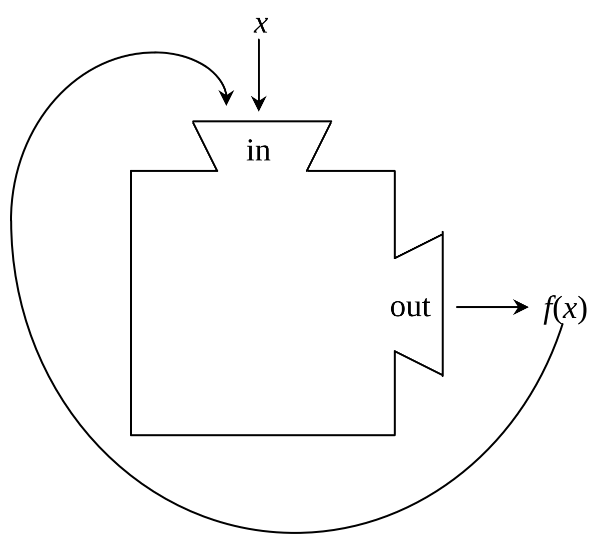


Figure 2

EXAMPLE Write the orbit of 16 for $f(x) = \sqrt{x}$.

First iterate: $x = 16$

Second iterate: $f(16) =$

Third iterate: $f(f(16)) =$

Fourth iterate: $f(f(f(16))) =$

Fifth iterate: $f(f(f(f(16)))) =$

EXAMPLE Write the orbit of 0 for $f(x) = \sqrt{1+x^2}$.

First iterate: $x = 0$

Second iterate: $f(0) =$

Third iterate: $f(f(0)) =$

Fourth iterate: $f(f(f(0))) =$

Fifth iterate: $f(f(f(f(0)))) =$

EXAMPLE Write the orbit of 2 for $f(x) = 1 + \frac{1}{x}$.

First iterate: $x = 2$

Second iterate: $f(2) =$

Third iterate: $f(f(2)) =$

Fourth iterate: $f(f(f(2))) =$

Fifth iterate: $f(f(f(f(2)))) =$

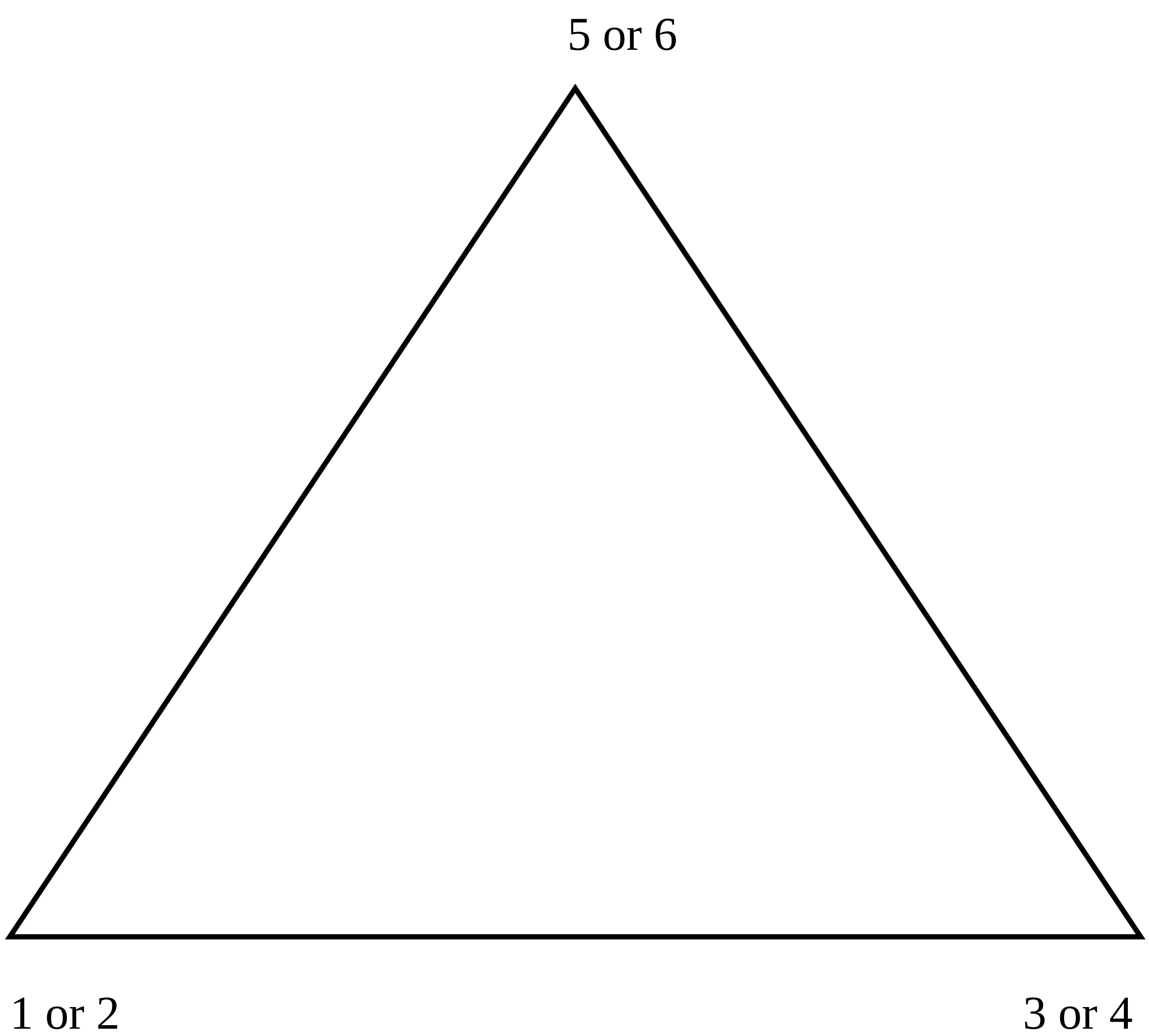
The Chaos Game - Lines

Step 1. Pick any point within the triangle as the starting point.

Step 2. Roll a die. The number that appears will indicate which vertex you are to move toward.

Step 3. Draw a line segment from your starting point to the point that is halfway to the indicated vertex.

Step 4. The point halfway to the vertex is your new starting point. Go to Step 2 and repeat the process.



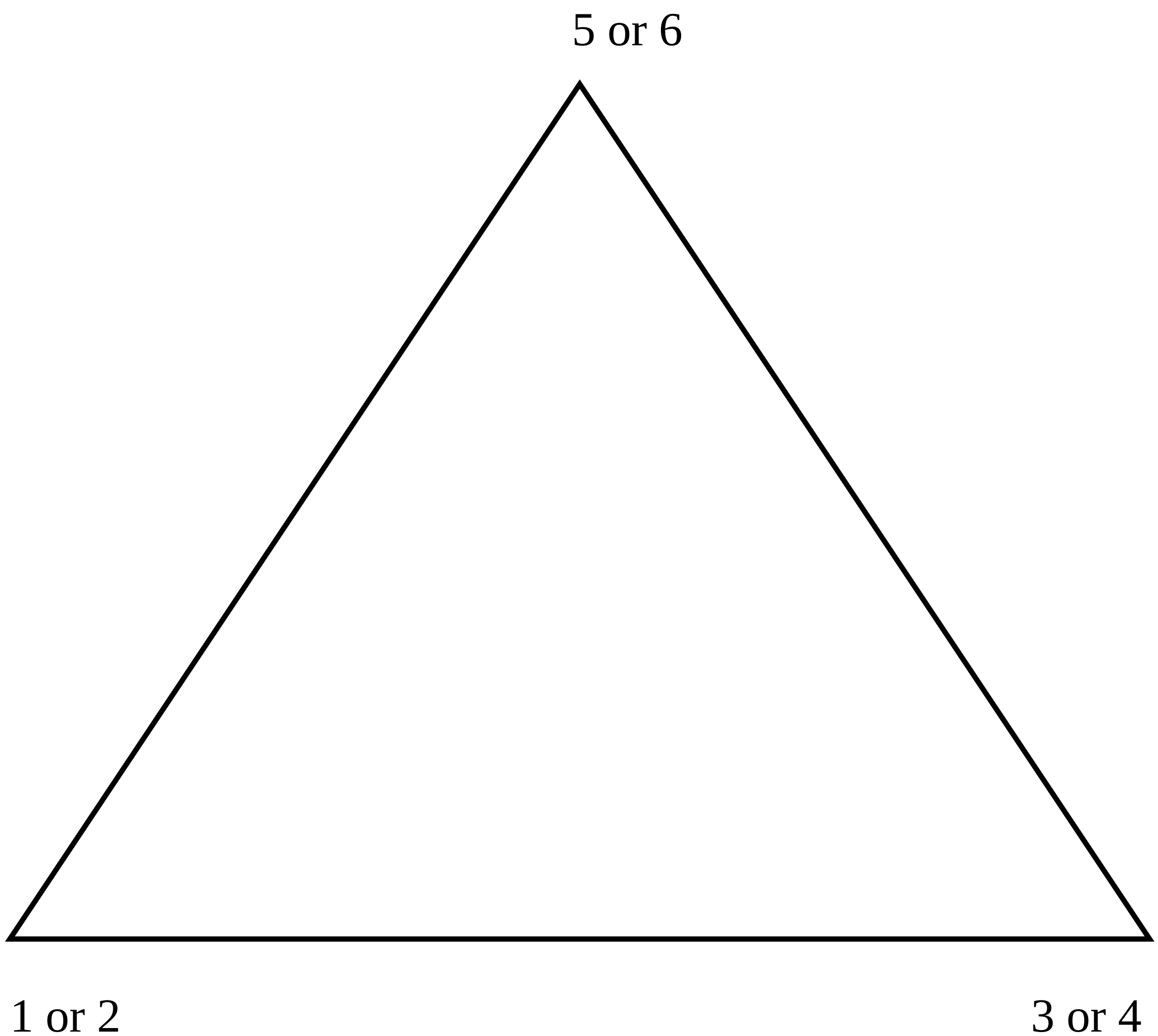
The Chaos Game - Dots

Step 1. Pick any point within the triangle as the starting point.

Step 2. Roll a die. The number that appears will indicate which vertex you are to move toward.

Step 3. Place a dot at the point that is halfway to the indicated vertex.

Step 4. The point halfway to the vertex is your new starting point. Go to Step 2 and repeat the process.



Chaos Game on the TI Graphing Calculator

The program below plays the Chaos game with line segments. It is just a matter of changing one line to get the Chaos game with points instead of line segments. When I do this is class, I have both versions of the game programmed into the calculator.

Note that I use variables C and D instead of X and Y in the program. When I use Y in place of D, I get some lines and points that seem like they are wrong.

I'm not sure if all of these first four lines are necessary.

```
:FnOff
:PlotsOff
:GridOff
:ClrDraw
:0→Xmin
:1→Xmax
:0→Ymin
:1→Ymax
```

You can replace these four lines with .5→C and .5→D if you want the program to run without inputting the first point.

```
:Disp "X="
:Input C
:Disp "Y="
:Input D
:Pt-On(C,D)
:For(K,1,1500)
:C→A
:D→B
:randInt(1,3) →N
:If N=1
:Then
:.5C→C
:.5D→D
:End
:If N=2
:Then
:.5(C+1)→C
:.5D→D
:End
:If N=3
:Then
:.5(C+.5)→C
:.5(D+1)→D
:End
:Pt-On(C,D)
:Line(A,B,C,D)
:End
```

Delete the :Line(A,B,C,D) line to get just points instead of lines.